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Folded three-spin string solutions in $AdS_5 \times S^5$

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Abstract: We construct a spinning closed string solution in $AdS_5 \times S^5$ which is folded in the radial direction and has two equal spins in AdS_5 and a spin in S^5. The energy expression of the three-spin solution specified by the folding and winding numbers for the small S^5 spin shows a logarithmic behavior and a one-third power behavior of the large total AdS_5 spin, in the long string and in the short string located near the boundary of AdS_5 respectively. It exhibits the non-regular expansion in the 't Hooft coupling constant, while it takes the regular one when the S^5 spin becomes large.

Keywords: Bosonic Strings, AdS-CFT and dS-CFT Correspondence, Penrose limit and pp-wave background.

Contents

1. Introduction

The AdS/CFT correspondence [1]–[3] has been studied in the supergravity approximation. In order to verify the correspondence in its full extent it is desirable to go beyond the low energy supergravity approximation. Under the circumstance that it is hard to quantize the superstring theory in $AdS_5 \times S^5$, the solvability of the string theory in the pp-wave background [4, 5] has opened a new step and led to an interesting proposal identifying particular stringy oscillator states with gauge invariant near-BPS operators with large R-charge in the BMN limit for the $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory [6]. This proposal has been interpreted as a special case of a semiclassical expansion of $AdS_5 \times S^5$ string theory selecting a particular sector of states where the string shrinks to a point and moves with a large angular momentum along a large circle of S^5 [7] and further developed [8]–[16]. A considerable amount of work has followed for various semiclassical string or membrane solutions [17]–[29].

Motivated by attempts to AdS/CFT correspondence to non-BPS states there have been the constructions of semiclassical string solutions with several angular momenta in different directions of AdS_5 and S^5. A novel solution describing a circular closed string rotating simultaneously in two planes in AdS_5 with equal spins S has been shown to produce a large-spin expansion of the energy whose subleading term scales with spin as $S^{1/3}$ instead of $\log S$ in the one-spin solution, when the point-like string is located close to the boundary of AdS_5 [30]. There have been the other solutions describing a circular closed string with three SO(6) angular momenta in S^5 [30, 31] and a folded closed string with two SO(6) angular momenta [32] located at the origin of AdS_5. A folded closed string solution with three spins (J_1, J_2, J_3) in S^5 has been constructed as a periodic solution of a Neumann one-dimensional integrable system [33], where the string is stretched along two angular directions and bent at one point. A three-spin string solution of circular type has been also analyzed and a two-spin one has been explicitly represented by the elliptic functions and a winding number. Their space-time energies expressed by the spins and the string tension $\sqrt{\lambda}$ are arranged to take the regular expansions in the 't Hooft coupling constant λ and remarkably matched onto the associated scaling dimensions of the non-BPS operators obtained by analyzing the relevant Bethe equation for a spin chain model in the perturbative SYM theory [32]–[39].

A hybrid (S, J) state represented by a string rotating in one plane in S^5 and having also one large spin in AdS_5 has been shown to be analytically continued to the (J_1, J_2) state represented by a folded string rotating in two planes in S^5 [37]. A more general solution describing a string rotating in both AdS_5 and S^5 with constant radii has been constructed as a periodic solution of a Neumann-Rosochatius one-dimensional integrable system to be characterized by the $2+3$ spins (S_a, J_i) and the $2+3$ winding numbers [40]. Its space-time energy has a regular large-spin expansion if there is at least one large spin in S^5. There have been various investigations about the multi-spin string solutions and the gauge-string duality [41]–[51]. In ref. [50] a spinning solution folded in the radial direction with two equal spins in AdS_5 has been presented, where the string is orbiting around the origin and the radius of string is not constant but oscillates between two locations. This solution specified by the folding and winding numbers associated with the radial direction and an angular direction of S^3 within AdS_5 shows a logarithmic behavior of the energy-spin relation for the long string.

We will construct a folded three-spin solution which has not only the two equal spins in AdS_5 but also a spin in S^5 and is stretched along both the radial and angular directions. It will be shown how this hybrid three-spin solution interpolates between the two-spin solution with the logarithmic energy-spin relation for the long string and that with the $S^{1/3}$ relation for the point-like string.

2. Folded three-spin solutions

The bosonic part of the type IIB superstring action for the $AdS_5 \times S^5$ background is given by

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\xi \left[G^{(AdS_5)}_{mn}(x)\partial_a x^m \partial^a x^n + G^{(S^5)}_{pq}(y)\partial_a y^p \partial^a y^q\right], \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'}, \tag{2.1}$$

where the $AdS_5 \times S^5$ metric in the global coordinates is expressed as

$$\begin{aligned}(ds^2)_{AdS_5} &= d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2)\,,\\ (ds^2)_{S^5} &= d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma(d\psi^2 + \cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2)\,.\end{aligned} \tag{2.2}$$

We consider a configuration that a closed string is spinning in the ϕ_1 and ϕ_2 directions of AdS_5 with equal spin as well as in the φ_3 direction of S^5, and stretched along the radial coordinate ρ and along the angular coordinate θ of S^5. We make the following ansatz describing this configuration

$$\begin{aligned}&t = \kappa\tau\,, \qquad \phi_1 = \omega\tau\,, \qquad \phi_2 = \omega\tau\,, \qquad \varphi_3 = \nu\tau\,,\\ &\rho = \rho(\sigma) = \rho(\sigma + 2\pi)\,, \qquad \theta = \theta(\sigma) = \theta(\sigma+2\pi)\,,\\ &\gamma = \psi = \varphi_1 = \varphi_2 = 0\,.\end{aligned} \tag{2.3}$$

The string equations of motion for θ and ρ are given by

$$(\sinh^2\rho\theta')' = 0\,, \qquad \rho'' - \sinh\rho\cosh\rho(\kappa^2 + \theta'^2 - \omega^2) = 0\,, \tag{2.4}$$

which lead to

$$\theta' = \frac{c}{\sinh^2\rho}\,, \qquad c = \text{const}\,, \tag{2.5}$$

$$\rho'' - c^2\frac{\cosh\rho}{\sinh^3\rho} - \frac{1}{2}(\kappa^2 - \omega^2)\sinh 2\rho = 0\,. \tag{2.6}$$

The conformal gauge constraint is described by

$$\rho'^2 = \kappa^2\cosh^2\rho - \frac{c^2}{\sinh^2\rho} - \omega^2\sinh^2\rho - \nu^2\,, \tag{2.7}$$

which is also the first integral of (2.6) with an appropriate integral constant. In terms of $x = \cosh\rho$ it can be rewritten by

$$x'^2 = (\omega^2 - \kappa^2)(a_+ - x^2)(x^2 - a_-)\,, \tag{2.8}$$

where the constants $a_\pm$ are

$$a_\pm = \frac{1}{2(\omega^2 - \kappa^2)}\left[2\omega^2 - \kappa^2 - \nu^2 \pm \sqrt{\kappa^4 + 2(2c^2 - \nu^2)\kappa^2 + \nu^4 - 4\omega^2c^2}\right]. \tag{2.9}$$

We impose a condition

$$1 \le a_- < a_+ \tag{2.10}$$

so that the string has two different turning points a_+, a_- in the radial direction. The integral of motion c and the parameter κ that is associated with the energy of the system, are described in terms of the turning points a_+, a_- as

$$c^2 = (\omega^2 - \kappa^2)(a_+ - 1)(a_- - 1)\,, \tag{2.11}$$

$$\kappa^2 = \omega^2 - \frac{\omega^2 - \nu^2}{a_+ + a_- - 1}\,, \tag{2.12}$$

which imply that the condition (2.10) yields $\nu < \kappa < \omega$. Therefore it follows that $a_- = 1$ for $c = 0$.

We choose a differential equation $x' = -[(\omega^2 - \kappa^2)(a_+ - x^2)(x^2 - a_-)]^{1/2}$ for (2.8), which is transformed through the change of variables $x/\sqrt{a_+} = y$, $\sqrt{a_+(\omega^2 - \kappa^2)}\sigma = u$ into

$$\frac{d}{du}y(u) = -\sqrt{(1 - y^2)\left(y^2 - \frac{a_-}{a_+}\right)}. \tag{2.13}$$

This expression yields a solution expressed by the Jacobi elliptic function

$$\begin{aligned}\cosh\rho &= \sqrt{a_+}\mathrm{dn}(\sqrt{a_+(\omega^2 - \kappa^2)}\sigma, m)\,,\\ &= \left(a_+ - (a_+ - a_-)\mathrm{sn}^2\left(\sqrt{a_+(\omega^2 - \kappa^2)}\sigma, m\right)\right)^{1/2}\,, \qquad m = \frac{a_+ - a_-}{a_+}.\end{aligned} \tag{2.14}$$

When $a_\pm$ are parametrized as $a_\pm = \cosh^2\rho_\pm$, from this expression we see that ρ indeed oscillates between ρ_+ and ρ_-. The periodicity condition $\rho(\sigma) = \rho(\sigma + 2\pi)$ is expressed in terms of a folding number N as

$$\frac{2\pi}{N}\sqrt{a_+(\omega^2 - \kappa^2)} = 2K(m) \tag{2.15}$$

since $\mathrm{dn}(u)$ and $\mathrm{sn}^2(u)$ in (2.14) have a fundamental period $2K$ where K is the complete elliptic integral of the first kind. In this string configuration composed of $2N$ segments ρ starts at ρ_+ and becomes ρ_- as σ goes from 0 to π/N for one segment. Alternatively the periodicity condition (2.15) can be derived by the direct integration of (2.8) as

$$\int_0^{2\pi} d\sigma = 2N\int_{\sqrt{a_+}}^{\sqrt{a_-}} dx \frac{-1}{\sqrt{(\omega^2 - \kappa^2)(a_+ - x^2)(x^2 - a_-)}}\,. \tag{2.16}$$

Substituting the solution (2.14) into (2.5) and integrating we have

$$\theta(\sigma) = \sqrt{\frac{a_- - 1}{a_+(a_+ - 1)}}\,\Pi\left(\sqrt{a_+(\omega^2 - \kappa^2)}\sigma, \frac{a_+ m}{a_+ - 1}, m\right), \qquad m = \frac{a_+ - a_-}{a_+}\,, \tag{2.17}$$

where $\Pi(u, n, m)$ is the elliptic integral of the third kind and we have chosen an integration constant such that $\theta(0) = 0$. From the periodicity condition $\theta(2\pi) = \theta(0) + 2\pi M$ for the angular coordinate the winding number M is specified as

$$2\pi M = N\theta\left(\frac{2\pi}{N}\right) = 2N\theta\left(\frac{\pi}{N}\right), \tag{2.18}$$

which combines with (2.17) and (2.15) to yield

$$\pi\frac{M}{N} = \sqrt{\frac{a_- - 1}{a_+(a_+ - 1)}}\,\Pi\left(\frac{a_+ - a_-}{a_+ - 1}, m\right), \tag{2.19}$$

where $\Pi(n, m) \equiv \Pi(K, n, m)$. Alternatively it follows from the direct integration of (2.5) together with (2.8)

$$\int_0^{2\pi M} d\theta = 2N\int_{\sqrt{a_+}}^{\sqrt{a_-}} dx \frac{-1}{\sqrt{(\omega^2 - \kappa^2)(a_+ - x^2)(x^2 - a_-)}}\frac{c}{x^2 - 1}. \tag{2.20}$$

The integers N, M will label different spinning configurations.

3. Energy-spin relations

The energy and three spins of spinning string solution are given by

$$\begin{aligned}
E &= \sqrt{\lambda}\kappa\int_0^{2\pi}\frac{d\sigma}{2\pi}\cosh^2\rho \equiv \sqrt{\lambda}\mathcal{E}\,, \qquad J = \sqrt{\lambda}\nu\,, \\
S_1 &= \sqrt{\lambda}\omega\int_0^{2\pi}\frac{d\sigma}{2\pi}\sinh^2\rho\cos^2\theta \equiv \sqrt{\lambda}\mathcal{S}_1\,, \\
S_2 &= \sqrt{\lambda}\omega\int_0^{2\pi}\frac{d\sigma}{2\pi}\sinh^2\rho\sin^2\theta \equiv \sqrt{\lambda}\mathcal{S}_2\,,
\end{aligned} \tag{3.1}$$

where J, S_1 and S_2 are the spins coming from the rotations in the φ_3, ϕ_1 and ϕ_2 directions, respectively. There is a relation between them

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}}{\omega} = 1\,, \qquad \mathcal{S} \equiv \mathcal{S}_1 + \mathcal{S}_2\,. \tag{3.2}$$

In view of (2.16) the energy is computed as

$$\mathcal{E} = 2N\kappa \int_{\sqrt{a_+}}^{\sqrt{a_-}} \frac{dx}{2\pi} \frac{-x^2}{\sqrt{(\omega^2-\kappa^2)(a_+-x^2)(x^2-a_-)}} = \frac{N\kappa\sqrt{a_+}}{\pi\sqrt{\omega^2-\kappa^2}} E(m)\,, \tag{3.3}$$

where $E(m)$ is the complete elliptic integral of the second kind. The substitution of κ in (2.12) into (3.3) provides

$$\mathcal{E} = \frac{N\sqrt{a_+}}{\pi}\left(\frac{\nu^2+\omega^2(a_++a_--2)}{\omega^2-\nu^2}\right)^{1/2} E(m)\,. \tag{3.4}$$

From (2.15) and (2.12), the angular velocity parameter ω is given by

$$\omega = \left(\nu^2 + \frac{a_++a_--1}{\pi^2 a_+}K(m)^2N^2\right)^{1/2} \tag{3.5}$$

so that the energy expression (3.4) is expressed as

$$\mathcal{E} = \frac{a_+E}{K}\left(\nu^2 + \frac{a_++a_--2}{\pi^2 a_+}K^2N^2\right)^{1/2}. \tag{3.6}$$

On the other hand we use (3.5) and (2.12) to rewrite the relation (3.2) as

$$\mathcal{E} = \left(1 + \frac{\mathcal{S}}{\sqrt{\nu^2+\frac{a_++a_--1}{\pi^2a_+}K^2N^2}}\right)\left(\nu^2 + \frac{a_++a_--2}{\pi^2 a_+}K^2N^2\right)^{1/2}. \tag{3.7}$$

The comparison between (3.6) and (3.7) yields the total AdS_5 spin

$$\mathcal{S} = \left(\frac{a_+E}{K} - 1\right)\left(\nu^2 + \frac{a_++a_--1}{\pi^2 a_+}K^2N^2\right)^{1/2}. \tag{3.8}$$

In order to analyze the above equations it is convenient to use two parameters $m = (a_+ - a_-)/a_+, n = (a_+-a_-)/(a_+-1)$ instead of the turning points a_+, a_-. Thus from (3.6), (3.8) and (2.19) we have the following three equations

$$\mathcal{E} = \frac{nE(m)}{n-m}\left(\frac{\nu^2}{K(m)^2} + \frac{m(2-n)}{\pi^2 n}N^2\right)^{1/2}, \tag{3.9}$$

$$\mathcal{S} = \left(\frac{nE(m)}{n-m} - K(m)\right)\left(\frac{\nu^2}{K(m)^2} + \frac{n+m-nm}{\pi^2 n}N^2\right)^{1/2}, \tag{3.10}$$

$$\pi\frac{M}{N} = \sqrt{\frac{(1-n)(n-m)}{n}}\Pi(n,m)\,, \tag{3.11}$$

which are expressed in terms of the auxiliary two independent parameters m, n so that if m, n are eliminated from the three transcendental equations, the energy E is in principle derived as a function of S, J, M and N. The relation (3.7) is also expressed in terms of m, n as

$$\mathcal{E} = \left(1 + \frac{\mathcal{S}}{\sqrt{\nu^2 + \frac{n+m-nm}{\pi^2 n} K(m)^2 N^2}} \right) \left(\nu^2 + \frac{m(2-n)}{\pi^2 n} K(m)^2 N^2 \right)^{1/2} . \tag{3.12}$$

Now we are ready to extract explicit analytic expressions of energy-spin relations in several special regions specified by J or $S \gg \sqrt{\lambda}$ and J or $S \ll \sqrt{\lambda}$. The condition (2.10) is translated into $0 < m < n \le 1$. The turning points ρ_+, ρ_- are written in terms of m, n as

$$\cosh^2 \rho_+ = \frac{n}{n-m}, \qquad \cosh^2 \rho_- = \frac{n(1-m)}{n-m} . \tag{3.13}$$

In view of $\cosh^2 \rho_+ - \cosh^2 \rho_- = m/(1 - \frac{m}{n})$, a parameter region $n \approx m \approx 1$ corresponds to the long string, while a region specified by $m/n \ll 1$ and $m \ll 1$ to the short string. There is the other interesting short string region $n \approx m \ll 1 - \frac{m}{n} \ll 1$.

We start to consider the long string case and write down a formula

$$\begin{aligned} \Pi(n,m) \; &= K(m) + \frac{2\sqrt{1 - k'^2 \sin^2 \psi}}{k'^2 \sin 2\psi} \times \\ &\times \left[F(\psi, k'^2) K(m) - E(\psi, k'^2) K(m) - F(\psi, k'^2) E(m) + \frac{\pi}{2} \right] , \end{aligned} \tag{3.14}$$

where $k'^2 = 1 - m, n = 1 - k'^2 \sin^2 \psi$, and $F(\psi, k'^2)$ and $E(\psi, k'^2)$ are the elliptic integrals of the first and second kind respectively. For $n \approx m \approx 1$ it approximately reduces to

$$\Pi(n,m) \approx \sqrt{\frac{n}{(1-n)(n-m)}} \left(\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{1-n}{1-m}} \right) , \tag{3.15}$$

whose substitution into (3.11) provides

$$\sqrt{\frac{1-n}{1-m}} \approx \cos \frac{M}{N} \pi . \tag{3.16}$$

We can now use (3.16) to rewrite (3.9) and (3.10) as

$$\mathcal{E} \approx \left(\frac{\nu^2}{K^2} + \frac{N^2}{\pi^2} \right)^{1/2} \frac{1}{(1-m) \sin^2 \left(\frac{M}{N} \pi \right)} \left(1 + \frac{1-m}{2} \log \frac{16}{1-m} \right) , \tag{3.17}$$

$$\mathcal{S} \approx \left(\frac{\nu^2}{K^2} + \frac{N^2}{\pi^2} \right)^{1/2} \frac{1}{(1-m) \sin^2 \left(\frac{M}{N} \pi \right)} \left[1 + \frac{1-m}{2} \left(1 - \sin^2 \left(\frac{M}{N} \pi \right) \right) \log \frac{16}{1-m} \right] \tag{3.18}$$

with $K(m) \approx \frac{1}{2} \log \frac{16}{1-m}$. Their difference gives

$$\mathcal{E} - \mathcal{S} \approx \frac{1}{2} \left(\frac{\nu^2}{K^2} + \frac{N^2}{\pi^2} \right)^{1/2} \log \frac{16}{1-m} . \tag{3.19}$$

For the region $\nu/\log(\frac{1}{1-m}) \ll 1$, the large total spin $\mathcal{S}$ is approximately estimated as

$$\mathcal{S} \approx \frac{N/\pi}{(1-m)\sin^2\left(\frac{M}{N}\pi\right)} \tag{3.20}$$

so that we have a logarithmic scaling behavior

$$\mathcal{E} - \mathcal{S} \approx \frac{N}{2\pi}\log \pi\mathcal{S} + \frac{\pi\nu^2}{N\log\pi\mathcal{S}} + \cdots . \tag{3.21}$$

Restoring the λ-dependence provides

$$E - S \approx \frac{N\sqrt{\lambda}}{2\pi}\log\frac{\pi S}{\sqrt{\lambda}} + \frac{\pi J^2}{N\sqrt{\lambda}\log\frac{\pi S}{\sqrt{\lambda}}} + \cdots, \qquad \frac{S}{\sqrt{\lambda}} \gg 1,\ \frac{J}{\sqrt{\lambda}} \ll \log\frac{S}{\sqrt{\lambda}}, \tag{3.22}$$

which shows a semiclassical $\sqrt{\lambda}$ expansion for the energy-spin relation. When $J = 0$ this expression reduces to the result of ref. [50].

For the opposite region $\nu/\log(\frac{1}{1-m}) \gg 1$, we have an approximate relation $\log(\mathcal{S}/\nu) \approx \log(1/(1-m))$, which yields

$$\mathcal{E} - \mathcal{S} \approx \nu + \frac{N^2}{8\pi^2\nu}\log^2\frac{\mathcal{S}}{\nu} + \cdots . \tag{3.23}$$

Thus we have a regular expansion in integer powers of λ

$$E - S \approx J + \frac{\lambda N^2}{8\pi^2 J}\log^2\frac{S}{J} + \cdots, \qquad \frac{S}{\sqrt{\lambda}} \gg 1,\ \frac{J}{\sqrt{\lambda}} \gg \log\frac{S}{J}. \tag{3.24}$$

Now we analyze the short string region specified by $m \ll n \ll 1$ that is more restricted than $m \ll n < 1$. Eq. (3.11) is approximately given by

$$\frac{2M}{N} \approx \sqrt{1 - \frac{m}{n}}, \tag{3.25}$$

which is satisfied when $2M < N$. The radial location of short string is so specified by

$$\cosh^2\rho_+ \approx \cosh^2\rho_- \approx \left(\frac{N}{2M}\right)^2 \tag{3.26}$$

that the string is located near the origin of AdS_5.

For $\nu \ll 1$ from (3.10) the total spin $\mathcal{S}$ is approximately estimated to be a small value

$$\mathcal{S} \approx \frac{N}{2}\frac{m}{n} \ll 1 \tag{3.27}$$

and then the energy $\mathcal{E}$ in (3.9) is specified by

$$\mathcal{E}^2 \approx \nu^2 + \mathcal{S}N . \tag{3.28}$$

This expression becomes

$$E^2 \approx J^2 + \sqrt{\lambda}NS, \qquad \frac{S}{\sqrt{\lambda}} \ll 1,\ \frac{J}{\sqrt{\lambda}} \ll 1, \tag{3.29}$$

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which shows the usual Regge trajectory relation when $J = 0$. The energy-spin relations of three-spin ($S_1 = S_2, J$) solution (3.22), (3.24) and (3.29) for the respective parameter regions are similar to the analogous relations of the two-spin (S, J) solution [8, 9].

For $\nu \gg 1$ eq. (3.10) can be approximately expressed as

$$\mathcal{S} \approx \frac{m}{n-m}\left(\nu^2 + \frac{N^2}{4}\left(1+\frac{m}{n}\right)\right)^{1/2} \tag{3.30}$$

from which it is noted that $\mathcal{S} \ll \nu$. Taking advantage of $\mathcal{S} \ll \nu$ to use an iterative expansion method we can derive a solution of (3.30) in a large-spin expansion form as

$$\frac{m}{n} \approx \frac{\mathcal{S}}{\nu} - \left(\frac{\mathcal{S}}{\nu}\right)^2 - \frac{N^2\mathcal{S}}{8\nu^3} + \frac{N^2\mathcal{S}^2}{8\nu^4} + \cdots . \tag{3.31}$$

Substituting this solution into the energy expression (3.12) instead of (3.9) we have

$$\mathcal{E} \approx \nu + \mathcal{S} + \frac{N^2\mathcal{S}}{8\nu^2} - \frac{N^2\mathcal{S}^2}{8\nu^3} + \cdots , \tag{3.32}$$

which reads

$$E \approx J + S + \frac{\lambda N^2 S}{8J^2} - \frac{\lambda N^2 S^2}{8J^3} + \cdots , \qquad \frac{J}{\sqrt{\lambda}} \gg 1,\ J \gg S . \tag{3.33}$$

The energy expression (3.33) for the three-spin solution in a near BMN limit $S/J \ll 1$ is compared with that for the two-spin (S, J) solution [8, 9, 37]. Here we write down the energy of the two-spin solution including full dependence on λ presented in [37]

$$E = J + S\sqrt{1 + \frac{\lambda}{J^2}} - \frac{\lambda S^2}{4J^3}\frac{2 + \lambda/J^2}{1 + \lambda/J^2} + \cdots , \tag{3.34}$$

whose expansion in λ/J^2

$$E \approx J + S + \frac{\lambda S}{2J^2} - \frac{\lambda S^2}{2J^3} + \cdots \tag{3.35}$$

has a similar behavior to (3.33). The analytic expansion in λ of (3.33) is also compared with the non-analytic one of the energy for (3.29).

Let us consider the parameter region $n \approx m \ll 1 - m/n \ll 1$ where the string becomes short and its radial location takes a large value $\cosh^2\rho_+^2 \approx \cosh^2\rho_-^2 \approx 1/(1 - m/n)$ so that the short string is close to the boundary of AdS_5. In this region the ratio M/N is specified by

$$\frac{2M}{N} \approx \sqrt{1 - \frac{m}{n}} \ll 1 . \tag{3.36}$$

For $\nu \ll 1$ eq. (3.10) is also expressed as (3.30), from which it is noted that $\mathcal{S} \gg 1$. Therefore the solution of (3.30) is approximately obtained by the iterative expansion procedure as

$$\frac{m}{n} \approx 1 - \frac{N}{\sqrt{2}\mathcal{S}} + \frac{5N^2}{8\mathcal{S}^2} - \frac{\nu^2}{\sqrt{2}N\mathcal{S}} + \cdots . \tag{3.37}$$

Combining this expansion with the energy expression (3.12), we derive

$$\mathcal{E} = \mathcal{S} + \frac{3\sqrt{2}N}{8} - \frac{11N^2}{64\mathcal{S}} + \frac{3\sqrt{2}\nu^2}{4N} + \cdots , \tag{3.38}$$

where the $\mathcal{S}\nu^2$ term has been canceled out. This expression yields a non-regular λ expansion

$$E = S + \frac{3\sqrt{2}\sqrt{\lambda}N}{8} - \frac{11\lambda N^2}{64S} + \frac{3\sqrt{2}J^2}{4\sqrt{\lambda}N} + \cdots , \qquad \frac{S}{\sqrt{\lambda}} \gg 1 \gg \frac{J}{\sqrt{\lambda}} . \tag{3.39}$$

Here from (3.36) N is much larger than M, while N is the same order as M in (3.29) and (3.33) for the previous cases. In order to trade large integer N for small finite interger M we use (3.36) and (3.37) to have $N \approx (2M)^{2/3}(\sqrt{2/\lambda}S)^{1/3}$, whose substitution into (3.39) for $J = 0$ yields

$$E = S + \frac{3}{4}\left(2M^2\lambda S\right)^{1/3} + \mathcal{O}\left(S^{-1/3}\right) . \tag{3.40}$$

In [40, 30] the energy of the circular constant-radii string solution with the large two spins S_1, S_2 in AdS_5 was presented as

$$E = S + \frac{3}{4}(\lambda S)^{1/3}\left(2k_1^2\frac{S_1}{S_2}\right)^{1/3} + \cdots , \tag{3.41}$$

where $k_1S_1 + k_2S_2 = 0, S = S_1 + S_2$ and $S_a \gg \sqrt{\lambda}$. When the two spins are equal, the two winding numbers k_a satisfy $k_1 = -k_2$ and the regular λ expansion (3.41) agrees with (3.40). Thus the short three-spin string with large S, when $J = 0$, corresponds to the circular point-like two-spin string located near the boundary of AdS_5.

For $\nu \gg 1$ the solution of (3.30) takes an expansion form

$$\frac{m}{n} \approx 1 - \frac{\nu}{\mathcal{S}} + \left(\frac{\nu}{\mathcal{S}}\right)^2 - \frac{N^2}{4\mathcal{S}\nu} + \frac{5N^2}{8\mathcal{S}^2} + \cdots \tag{3.42}$$

owing to $\mathcal{S} \gg \nu$. The energy expression (3.12) together with (3.42) yields

$$\mathcal{E} \approx \mathcal{S} + \nu + \frac{N^2}{8\nu} - \frac{N^2}{8\mathcal{S}} + \cdots , \tag{3.43}$$

where the $\mathcal{S}/\nu^2$ term has been canceled out. Thus restoring the λ-dependence we have

$$E \approx S + J + \frac{\lambda N^2}{8J} - \frac{\lambda N^2}{8S} + \cdots , \qquad \frac{S}{\sqrt{\lambda}} \gg \frac{J}{\sqrt{\lambda}} \gg 1 , \tag{3.44}$$

which shows a regular λ expansion. Using $N^2 \approx 4M^2S/J$ that is obtained from (3.36) and (3.42) we trade large integer N for small finite integer M to have

$$E = S + J + \frac{\lambda M^2 S}{2J^2} - \frac{\lambda M^2}{2J} + \cdots . \tag{3.45}$$

Its λ dependence remains the same as (3.44), while the $\lambda^{1/2}$ dependence of the subleading term in (3.39) is changed into the $\lambda^{1/3}$ dependence of the corresponding term in (3.40).

4. Conclusion

Analyzing the conformal gauge constraint and the closed string periodicity conditions for both the radial and angular directions in AdS_5, we have constructed a solution describing a folded three-spin string with two equal spins in AdS_5 and a spin in S^5. This string is orbiting around the origin and its energy is a function of the total AdS_5 spin S, the S^5 spin J and the folding and winding numbers as a parametric solution of the system of three transcendental equations.

We have observed that there exist three types of string configurations in certain limits; a long string with a large total spin S, a short string with a small S located near the origin of AdS_5 and a short string with a large S located near the boundary of AdS_5. By means of the iterative expansion method we have extracted the energy-spin relations for the short strings. For the large value of S there appear two different configurations such as the long string and the latter short string, whose energy expressions include the linear term of S irrespective of the magnitude of J. The former short string is characterized by the folding and winding numbers N, M with same order magnitude, while the latter short string by the small value of M/N. The long string as well as the former short string has been shown to produce the same enegy-spin relations as the two-spin (S, J) string, where a logarithmic behavior, a Regge-type one and a BMN-type one appear. We have demonstrated that the latter short string with $J = 0$ reproduces the energy-spin relation with a term $S^{1/3}$ for the two-spin string with two equal AdS_5 spins. We have observed that the energy expression for the latter short string with $S \gg \sqrt{\lambda} \gg J$ shows a non-regular λ expansion, while that with $S \gg J \gg \sqrt{\lambda}$ is indeed arranged to take a regular λ expansion. The non-perturbative λ expansion for the small S^5 spin is changed into the perturbative λ expansion for the large S^5 spin, which transition is also seen in the long string as well as the former short string. Thus the spin in S^5 should be large in order to have the regular λ expansion for the energy of folded three-spin solution, which includes the leading linear term of J. It is desirable to derive these regular λ expansions of the semiclassical string energy by computing the quantum anomalous dimension of the relevant gauge invariant operator including covariant derivatives from the SYM perturbation theory.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[5] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

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[6] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[7] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[8] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226].

[9] J.G. Russo, *Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 038 [hep-th/0205244].

[10] A. Armoni, J.L.F. Barbon and A.C. Petkou, *Orbiting strings in AdS black holes and $N = 4$ SYM at finite temperature*, *J. High Energy Phys.* **06** (2002) 058 [hep-th/0205280].

[11] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103].

[12] M. Alishahiha and M. Ghasemkhani, *Orbiting membranes in M-theory on $AdS_7 \times S^4$ background*, *J. High Energy Phys.* **08** (2002) 046 [hep-th/0206237].

[13] J.A. Minahan, *Circular semiclassical string solutions on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 648** (2003) 203 [hep-th/0209047].

[14] A.A. Tseytlin, *Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond*, *Int. J. Mod. Phys.* **A 18** (2003) 981 [hep-th/0209116].

[15] A. Armoni, J.L.F. Barbon and A.C. Petkou, *Rotating strings in confining AdS/CFT backgrounds*, *J. High Energy Phys.* **10** (2002) 069 [hep-th/0209224].

[16] M. Alishahiha and A.E. Mosaffa, *Circular semiclassical string solutions on confining AdS/CFT backgrounds*, *J. High Energy Phys.* **10** (2002) 060 [hep-th/0210122].

[17] S.A. Hartnoll and C. Núñez, *Rotating membranes on G_2 manifolds, logarithmic anomalous dimensions and $N = 1$ duality*, *J. High Energy Phys.* **02** (2003) 049 [hep-th/0210218].

[18] R.C. Rashkov and K.S. Viswanathan, *Rotating strings with B-field*, hep-th/0211197.

[19] A. Loewy and Y. Oz, *Large spin strings in AdS_3*, *Phys. Lett.* **B 557** (2003) 253 [hep-th/0212147].

[20] J.M. Pons and P. Talavera, *Semi-classical string solutions for $N = 1$ SYM*, *Nucl. Phys.* **B 665** (2003) 129 [hep-th/0301178].

[21] M. Alishahiha and A.E. Mosaffa, *Semiclassical string solutions on deformed NS5-brane backgrounds and new pp-wave*, hep-th/0302005.

[22] S. Ryang, *Rotating and orbiting strings in the near-horizon brane backgrounds*, *J. High Energy Phys.* **04** (2003) 045 [hep-th/0303237].

[23] H. Dimov, V. Filev, R.C. Rashkov and K.S. Viswanathan, *Semiclassical quantization of rotating strings in Pilch-Warner geometry*, *Phys. Rev.* **D 68** (2003) 066010 [hep-th/0304035].

[24] A.L. Larsen and M.A. Lomholt, *Open string fluctuations in AdS with and without torsion*, *Phys. Rev.* **D 68** (2003) 066002 [hep-th/0305034].

[25] D. Aleksandrova and P. Bozhilov, *On the classical string solutions and string/field theory duality*, *J. High Energy Phys.* **08** (2003) 018 [hep-th/0307113].

JHEP04(2004)053

[26] D. Aleksandrova and P. Bozhilov, *On the classical string solutions and string/field theory duality, II*, hep-th/0308087.

[27] M. Schvellinger, *Spinning and rotating strings for $N = 1$ SYM theory and brane constructions*, *J. High Energy Phys.* **02** (2004) 066 [hep-th/0309161].

[28] P. Bozhilov, *M2-brane solutions in $AdS_7 \times S^4$*, *J. High Energy Phys.* **10** (2003) 032 [hep-th/0309215].

[29] A. Khan and A.L. Larsen, *Spinning pulsating string solitons in $AdS_5 \times S^5$*, *Phys. Rev.* **D 69** (2004) 026001 [hep-th/0310019].

[30] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255].

[31] S. Frolov and A.A. Tseytlin, *Quantizing three-spin string solution in $AdS_5 \times S^5$*, *J. High Energy Phys.* **07** (2003) 016 [hep-th/0306130].

[32] S. Frolov and A.A. Tseytlin, *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [hep-th/0306143].

[33] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191].

[34] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **03** (2003) 013 [hep-th/0212208].

[35] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *J. High Energy Phys.* **09** (2003) 010 [hep-th/0306139].

[36] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[37] N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *J. High Energy Phys.* **10** (2003) 037 [hep-th/0308117].

[38] J. Engquist, J.A. Minahan and K. Zarembo, *Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$*, *J. High Energy Phys.* **11** (2003) 063 [hep-th/0310188].

[39] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.

[40] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, hep-th/0311004.

[41] A.V. Belitsky, A.S. Gorsky and G.P. Korchemsky, *Gauge/string duality for QCD conformal operators*, *Nucl. Phys.* **B 667** (2003) 3 [hep-th/0304028].

[42] L. Dolan, C.R. Nappi and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, *J. High Energy Phys.* **10** (2003) 017 [hep-th/0308089].

[43] A. Gorsky, *Spin chains and gauge/string duality*, hep-th/0308182.

[44] D. Mateos, T. Mateos and P.K. Townsend, *Supersymmetry of tensionless rotating strings in $AdS_5 \times S^5$ and nearly-BPS operators*, *J. High Energy Phys.* **12** (2003) 017 [hep-th/0309114].

[45] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *J. High Energy Phys.* **03** (2004) 004 [hep-th/0310182].

[46] A. Mikhailov, *Speeding strings*, *J. High Energy Phys.* **12** (2003) 058 [hep-th/0311019].

JHEP04(2004)053

[47] M. Kruczenski, *Spin chains and string theory*, hep-th/0311203.

[48] B. Stefanski Jr., *Open spinning strings*, *J. High Energy Phys.* **03** (2004) 057 [hep-th/0312091].

[49] N.W. Kim, *Multi-spin strings on $AdS_5 \times T^{1,1}$ and operators of $N = 1$ superconformal theory*, hep-th/0312113.

[50] A.L. Larsen and A. Khan, *Novel explicit multi spin string solitons in AdS_5*, hep-th/0312184.

[51] M. Alishahiha, A.E. Mosaffa and H. Yavartanoo, *Multi-spin string solutions in AdS black hole and confining backgrounds*, hep-th/0402007.

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